What is a quantum computer?

→ Quantum Mechanics
→ Quantum information theory
→ Computer Science

Quantum computers leverage specific properties described by quantum mechanics to perform computation

Classical computers leverage quantum mechanics as well, but not at computation level. Electron states in a classical computer gate behave semi-classically (in single particle picture). This is due to large number of scatterers the electron at valance and conduction band sees

Part 1 Recap of "Strangeness" of Quantum mechanics

If you think you understand quantum mechanics, then you don't

-Richard Feynmann

For example: Indistinguishability

electrons are identical they are indistinguishable
these are so strange quantum concepts

→ One-electron universe (John Wheeler)
All the electrons in the universe are the same electron, transversing time

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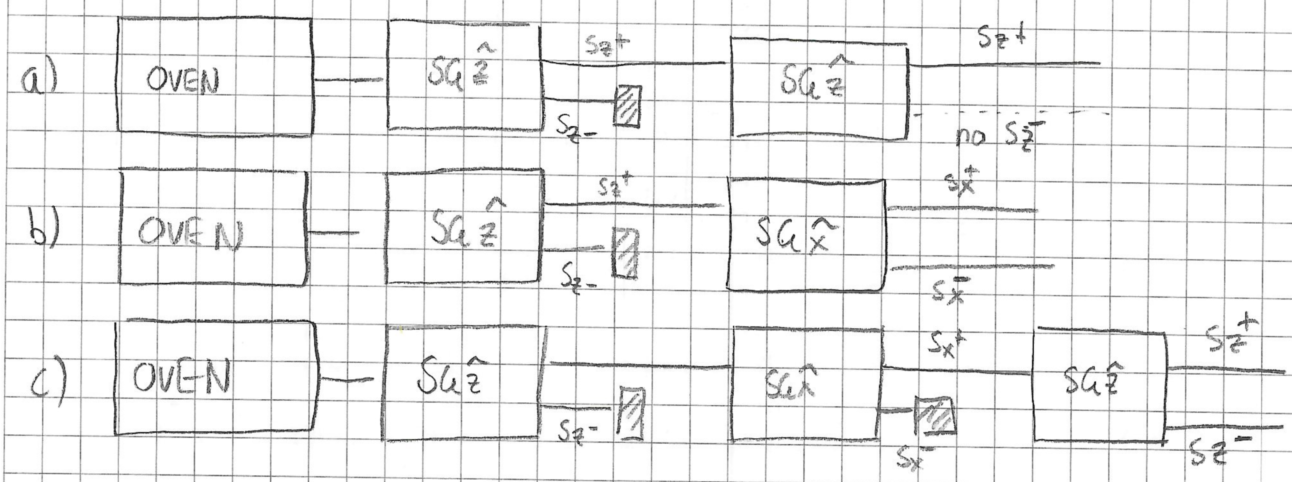
The experiment revealed only two possible values of the z-component of S ; S_z+ and S_z-. Numerically $\hbar/2$ and $-\hbar/2$

<u>Sequential Stern-Gerlach experiments</u>



setup in a) prooves we can select and detect S_z+ beam
setup in b), where we have a second SG apparatus which has an inhomogeneous field in $\hat{x}$ direction, the beam is again split into two S_x+ and S_x-. Does this mean S_z+ had %50 S_x+ and %50 S_x- inside? NO! Setup in c) puts the S_x+ beam into SG$\hat{z}$ apparatus again, and again there are S_z+ and S_z- components. This is inexplicible in classical sense.

$\rightarrow$ SG$\hat{x}$ destroys any previous information about S_z

If electron was really a spinning top

$$L = I\omega$$

we could've measured L_x and L_y easily. But since electron spin is quantum concept, we can not determine S_z and S_x simultaneously

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Silver has 47 electrons 46 of which is paired.
each silver atom has a magnetic moment μ proportional to spin magnetic moment of the 47th electron S

$$\mu \propto S \quad \left(\text{proportionality constant is } e/m_e c\right)$$
$$e<0$$

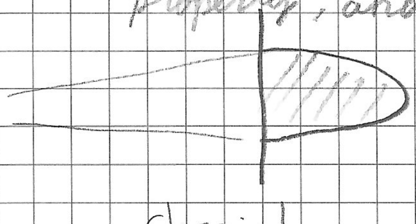
Force each atom experiences due to magnet

$$F_z = \frac{\partial}{\partial z}(\mu \cdot \vec{B}) \cong \mu_z \frac{\partial B_z}{\partial z}$$

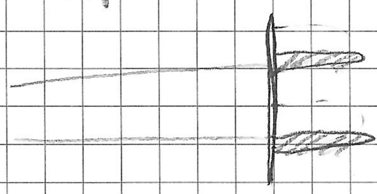
(where the field has been assumed homogeneous)
$\mu_z > 0$ $(S_z < 0)$ atom experiences a downward force
$\mu_z < 0$ $(S_z > 0)$ atom experiences an upward force

→ Stern - Gerlach apparatus measures the z-component of μ (or S) up to a proportionality factor

If electron spin was something like a classical spinning object, then the randomness in the oven (there is no preffered direction) would mean we can have anything between $-|\mu| \leq \mu_z \leq +|\mu|$. But, spin of electron is a quantum property, and the experiment result is "strange"



Classical expectation



strange result

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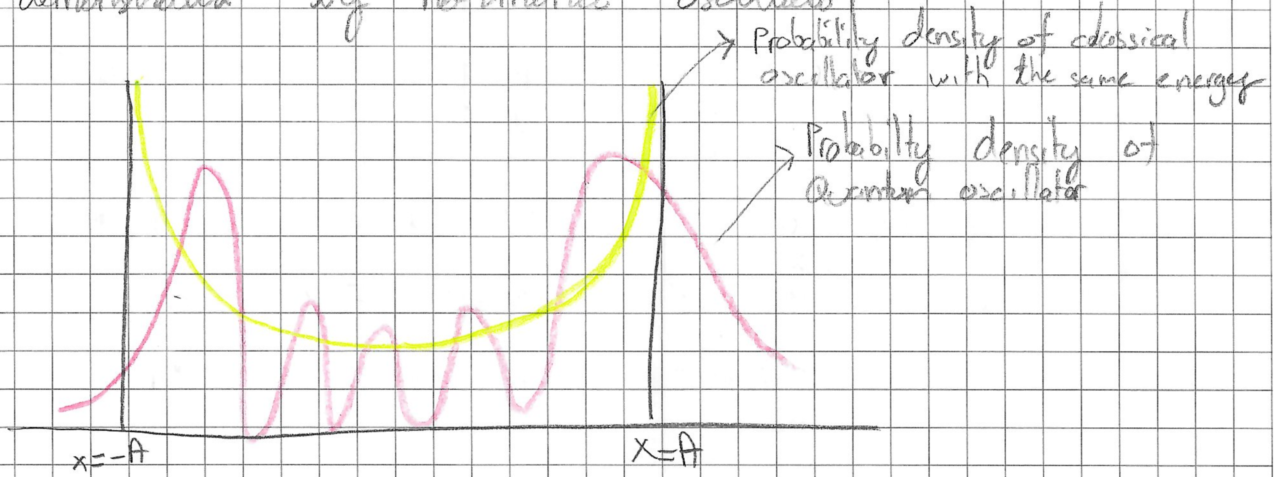
Feynmann "stole" the idea (his own words) in his 1949 paper "The theory of positrons"

Yochiro Nambu → there is no creation nor annihilation but only a change of directions of moving particles, from past to future or from future to past.

Probably the most relevant stark difference between a classical and a quantum system is demonstrated by harmonic oscillator



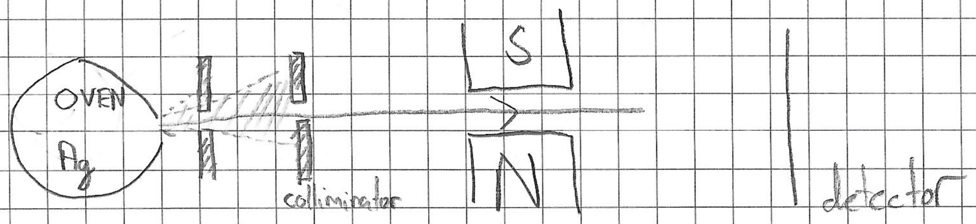
The Stern-Gerlach experiment

O. Stern (1921) → Conceived

W. Gerlach (1922) → Carried out in Frankfurt

Dramatic illustration for the necessity to radically depart from concepts of classical mechanics

Simplified model



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Analogy with Polarization of light

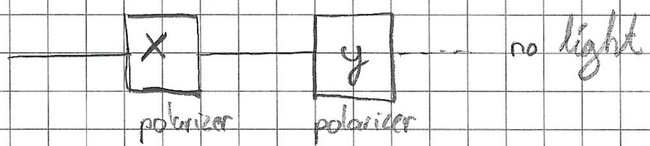
Consider a monochromatic light wave propagating in the z-direction

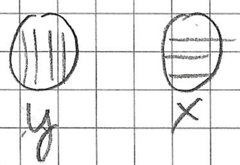
$$E = E_0 \hat{x} \cos(kz - wt)$$

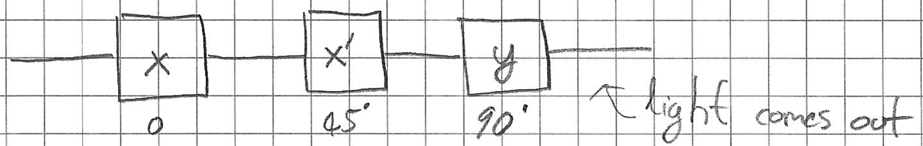
likewise for the y-polarized light

$$E = E_0 \hat{y} \cos(kz - wt)$$

One can obtain polarized light by using polarizers





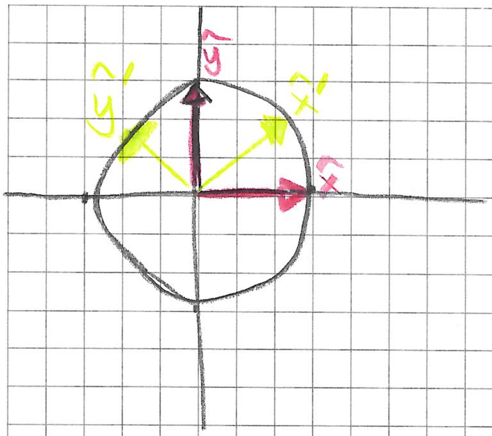


x' filter intervenes and destroys information about x filter

$$S_z \pm \text{ atoms} \leftrightarrow x-, y- \text{ polarized light}$$

$$S_x \pm \text{ atoms} \longleftrightarrow x'- \; y'- \text{ polarized light}$$

Let us examine how we can approach this situation algebraically



$$E_0 \hat{x}' \cos(kz-\omega t) = E_0 \left[\frac{1}{\sqrt{2}} \hat{x} \cos(kz-\omega t) + \frac{1}{\sqrt{2}} \hat{y} \cos(kz-\omega t) \right]$$

$$E_0 \hat{y}' \cos(kz-\omega t) = E_0 \left[-\frac{1}{\sqrt{2}} \hat{x} \cos(kz-\omega t) + \frac{1}{\sqrt{2}} \hat{y} \cos(kz-\omega t) \right]$$

- The light coming from the first polarizer is a combination of x'- and y'- polarized beam
- Second polarizer selects x'- polarized beam
- Third polarizer can select y- polarized beam since x'-polarized beam consists of x- and y- polarized beams

→ Analogy suggests we can utilize a two dimensional space to analytically describe the SG experiment
→ this is not xy space, it is abstract

S_x+ state can be described by a vector
we call this ket in Dirac notation

$$S_x^+ \longrightarrow |S_x;+\rangle$$
$$S_x^- \longrightarrow |S_x;-\rangle$$

following the analogy

$$|S_x;+\rangle \stackrel{?}{=} \frac{1}{\sqrt{2}} |S_z;+\rangle + \frac{1}{\sqrt{2}} |S_z;-\rangle$$

$$|S_x;-\rangle \overset{?}{=} -\frac{1}{\sqrt{2}}|S_z;+\rangle + \frac{1}{\sqrt{2}}|S_z;-\rangle$$

How are we going to represent the $S_y\pm$ states? We have already used up the available possibilities in writing $|S_x;\pm\rangle$, how can we distinguish $|S_y;\pm\rangle$ in terms of $|S_z;\pm\rangle$ from $|S_x;\pm\rangle$?

→ Circularly polarized light analogy

RCL $\quad E = E_0\left[\frac{1}{\sqrt{2}}\hat{x}\cos(kz-\omega t) + \frac{1}{\sqrt{2}}\hat{y}\cos(kz-\omega t+\pi/2)\right]$

$$Re(\mathcal{E}) = E/E_0$$

$$\mathcal{E} = \left[\frac{1}{\sqrt{2}}\hat{x}e^{i(kz-\omega t)} + \frac{i}{\sqrt{2}}\hat{y}e^{i(kz-\omega t)}\right] \quad (i = e^{i\pi/2})$$

then

$$|S_y;\pm\rangle \overset{?}{=} \frac{1}{\sqrt{2}}|S_z;+\rangle \pm \frac{i}{\sqrt{2}}|S_z;-\rangle$$

Analogy tells us!

→ We need a two dimensional space

→ This space has to be complex vector space!

The analogy becomes even more interesting when you consider photon (quantum) nature of light!

Ket Space

In quantum mechanics a physical state, for example, a silver atom with a definitive spin orientation is represented by a state vector in complex vector space. Following Dirac, we call this a "ket" and denote it by $|\alpha\rangle$. If the vector space spans a continious spectra, the space is called the Hilbert space.

Two kets can be added

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

Kets can be multiplied by complex scalars

$$c|\alpha\rangle = |\alpha\rangle c$$

when $c = 0+0i$ → resulting ket is the null ket

physics postulate → $|\alpha\rangle$ and $c|\alpha\rangle$ represent same physical state $c \neq 0$

Hence only the direction is important
Mathematicians prefer rays instead of vectors

An <u>observable</u>, such as momentum and spin components, can be represented by an <u>operator</u> such as A, in the vector space in question. Quite generally, an operator acts on a ket from the left

$$A \cdot (|\alpha\rangle) = A|\alpha\rangle$$

$A|\alpha\rangle$ is a new ket, not a constant × ket!

There are particular kets of importance called <u>eigenkets</u> of operator A which satisfy

$$A|a'\rangle = a'|a'\rangle, \quad A|a''\rangle = a''|a''\rangle \cdots$$

here

$$|a'\rangle, |a''\rangle, \dots \quad ; \{|a'\rangle\}$$

are called the eigenkets and

$$a', a'', a''', \dots$$

are called the eigenvalues

The physical state corresponding to an eigenket is called an eigenstate. for spin

$$S_z|S_z;+\rangle = \hbar/2\,|S_z;+\rangle, \quad S_z|S_z;-\rangle = -\tfrac{\hbar}{2}|S_z;-\rangle$$

Similarly

$$S_x|S_x;\pm\rangle = \pm\hbar/2\,|S_x;\pm\rangle$$

Any arbitrary ket can be decomposed into eigenkets

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$$

complex coefficient

Bra space and inner products

The "bra" space is dual to the ket space

The bra space is spanned by eigenbras $\{\langle a'|\}$

$$c_\alpha|\alpha\rangle + c_\beta|\beta\rangle \xleftrightarrow{DC} c_\alpha^*\langle\alpha| + c_\beta^*\langle\beta|$$

notice the complex conjugate

Since $\langle\beta|$ and $|\alpha\rangle$ are vectors, we can define vector operations between them. The inner product

$$\langle\beta|\alpha\rangle = (\langle\beta|)\cdot(|\alpha\rangle)$$

bra(c)ket

This gives you a complex scalar in return two fundemental properties

1. $$\boxed{\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*}$$

notice that this is different from scalar product where $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ since this is a complex vector space

2. $$\boxed{\langle\alpha|\alpha\rangle \geqslant 0}$$

This is sometimes called positive definitive metric as a physicist this is essential for probabilistic interpretation of quantum mechanics

Two kets are orthogonal if

$$\langle\alpha|\beta\rangle = 0 \equiv \langle\beta|\alpha\rangle = 0$$

Non null kets can form a normalized ket

$$|\tilde{\alpha}\rangle = \left(\frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}\right)|\alpha\rangle$$

and hence

$$\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1$$

$\sqrt{\langle\alpha|\alpha\rangle}$ is known as the norm of $|\alpha\rangle$

For probabilistic interpretation, it is very useful to have normalized kets

Operators

An operator acts on a ket from the left side

$$X.(|\alpha\rangle) = X|\alpha\rangle$$

and produces another ket

$X = Y$ when $X|\alpha\rangle = Y|\alpha\rangle$

X is a null operator when $X|\alpha\rangle = 0$

Operators can be added. Addition is commutative and associative in general

$$X+Y = Y+X$$

$$X+(Y+Z) = (X+Y)+Z$$

with the exception of time-reversal

A linear operator satisfies

$$X(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle) = c_\alpha X|\alpha\rangle + c_\beta X|\beta\rangle$$

An operator X always acts on a bra from the right side

$$(\langle\alpha|).X = \langle\alpha|X$$

resulting in another bra. $X|\alpha\rangle$ and $\langle\alpha|X$ are NOT NECESARILY dual to each other

$$X|\alpha\rangle \longleftrightarrow \langle\alpha|X^{+}$$

An operator is Hermitian if

$$X = X^{+}$$

Multiplication

Multiplication operations in general are non-commutative

$$\boxed{XY \neq YX}$$

They are associative

$$\boxed{X(YZ) = (XY)Z = XYZ}$$

$$X(Y|\alpha\rangle) = (XY)|\alpha\rangle = XY|\alpha\rangle$$

$$(\langle\beta|X)Y = \langle\beta|(XY) = \langle\beta|XY$$

Notice

$$(XY)^{\dagger} = Y^{\dagger}X^{\dagger}$$

Outer product

$$(|\beta\rangle)\cdot(\langle\alpha|) = |\beta\rangle\langle\alpha| \quad \text{(can be regarded as operator)}$$

The Associative axiom of multiplication

- $(\underbrace{|\beta\rangle\langle\alpha|}_{X})\cdot|\gamma\rangle = |\beta\rangle\cdot(\overbrace{\langle\alpha|\gamma\rangle}^{c})$

  $X = |\beta\rangle\langle\alpha|$ then $X^{\dagger} = |\alpha\rangle\langle\beta|$

- $(\langle\beta|)\cdot(X|\alpha\rangle) = (\langle\beta|X)\cdot(|\alpha\rangle)$

  $\rightarrow \langle\beta|X|\alpha\rangle$

- $\langle\beta|X|\alpha\rangle = \langle\beta|\cdot(X|\alpha\rangle) = \{(\langle\alpha|X^{*})|\beta\rangle\}^{*}$

  $= \langle\alpha|X^{\dagger}|\beta\rangle^{*}$

For Hermitian X we have

$$\boxed{\langle\beta|X|\alpha\rangle = \langle\alpha|X|\beta\rangle^*}$$

Base kets and Matrix representations

Theorem : The eigenvalues of a Hermitean operator A are real ; the eigenkets of A corresponding to different eigenvalues are orthogonal

$$A|a'\rangle = a'|a'\rangle \quad \rightarrow \text{ multiply by } \langle a''|$$

$$-\quad \langle a''|A = a''^* \langle a''| \quad \rightarrow \text{ multiply by } |a'\rangle$$

$$(a' - a''^*)\ \langle a''|a'\rangle = 0$$

if $|a'\rangle = |a''\rangle \rightarrow a' = a''^*$

if $|a'\rangle \neq |a''\rangle \rightarrow \langle a''|a'\rangle = 0$

Eigenkets as base kets

given an arbitrary ket

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$$

$$c_{a'} = \langle a'|\alpha\rangle$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

only works when

$$\boxed{\sum_{a'} |a'\rangle\langle a'| = 1} \quad \text{(completeness or closure)}$$

Closure relation is extremely useful! Consider

$$\langle\alpha|\alpha\rangle = \langle\alpha|.\mathbb{1}.|\alpha\rangle = \langle\alpha|.\Big(\sum_{a'}|a'\rangle\langle a'|\Big)|\alpha\rangle$$

$$= \sum_{a'} |\langle a'|\alpha\rangle|^2$$

hence

$$\sum_{a'} |c_{a'}|^2 = \sum_{a'} |\langle a'|\alpha\rangle|^2 = 1$$

$|a'\rangle\langle a'|$ is an outer product thus, it is an operator

$$(|a'\rangle\langle a'|)|\alpha\rangle = |a'\rangle(\langle a'|\alpha\rangle) = c_{a'}|a'\rangle$$

it selects the portion of $|\alpha\rangle$ parallel to $|a'\rangle$. This is called the projection operator along the base ket $|a'\rangle$ and is denoted by $\Lambda_{a'}$

$$\Lambda_{a'} = |a'\rangle\langle a'|$$

due to completeness

$$\boxed{\sum_{a'} \Lambda_{a'} = 1}$$

Matrix Representations

Having specified the base kets, we can now represent an operator, X, by a square matrix

$$X = \sum_{a''}\sum_{a'} |a''\rangle\langle a''|X|a'\rangle\langle a'|.$$

if the dimensionality of ket space is N, there are N^2 $\langle a''|X|a'\rangle$ elements. This can be represented as an $N \times N$ square matrix

$\langle a'' | X | a' \rangle$
row column

$$X \doteq \begin{pmatrix} \langle a^{(1)} | X | a^{(1)} \rangle & \langle a^{(1)} | X | a^{(2)} \rangle & \cdots \\ \langle a^{(2)} | X | a^{(1)} \rangle & \langle a^{(2)} | X | a^{(2)} \rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$\doteq$: is represented by
The operator is different from its representation similar to how an actress is different from the poster of the actress

$$\langle a'' | X | a' \rangle = \langle a' | X^{\dagger} | a'' \rangle^*$$

when an operator is Hermitean

$$\langle a'' | B | a' \rangle = \langle a' | B | a'' \rangle^*$$

$\langle a'' | X | a' \rangle$ arrangement as a square matrix is in conformity with $Z = XY$

The representation of X also sets representation of bra and ket space

$$|\gamma\rangle = X|\alpha\rangle$$

$$\langle a' | \gamma \rangle = \langle a' | X | \alpha \rangle$$

$$= \sum_{a''} \langle a' | X | a'' \rangle \langle a'' | \alpha \rangle$$

square matrix × column matrix

$$|\alpha\rangle \doteq \begin{pmatrix} \langle a^{(1)} | \alpha \rangle \\ \langle a^{(2)} | \alpha \rangle \\ \vdots \end{pmatrix} \qquad |\gamma\rangle \doteq \begin{pmatrix} \langle a^{(1)} | \gamma \rangle \\ \langle a^{(2)} | \gamma \rangle \\ \vdots \end{pmatrix}$$

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$$\langle \gamma | = \langle \alpha | X$$

$$\langle \gamma | a' \rangle = \sum_{a''} \langle \alpha | a'' \rangle \langle a'' | X | a' \rangle$$

row matrix × square matrix

$$\langle \gamma | \doteq \left(\langle \gamma | a^{(1)} \rangle, \langle \gamma | a^{(2)} \rangle, \ldots \right)$$

$$= \left(\langle a^{(1)} | \gamma \rangle^*, \langle a^{(2)} | \gamma \rangle^*, \ldots \right)$$

notice that these representations are consistent with

$$\langle \beta | \alpha \rangle = \text{(complex) number}$$

$$|\beta\rangle\langle\alpha| = \text{operator}$$

using rules of linear algebra

The matrix representation of an observable A using eigenkets of A is a diagonal matrix

$$A = \sum_{a'} a' |a'\rangle\langle a'| = \sum_{a'} a' \Lambda_{a'}$$

P.A.M. Dirac : "A measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured"

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$|\alpha\rangle \xrightarrow{\text{A measurement}} |a'\rangle$$

$$|a'\rangle \xrightarrow{\text{A measurement}} |a'\rangle$$

Then the probability of finding the a' state is

$$a' = |\langle a' | \alpha \rangle|^2$$ ($|\alpha\rangle$ must be normalised)

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Expectation value of A taken with respect to state $|\alpha\rangle$ is defined as

$$\langle A\rangle \equiv \langle\alpha|A|\alpha\rangle = \langle A\rangle_\alpha$$

$$\langle A\rangle = \sum_{a'}\sum_{a''} \langle\alpha|a''\rangle\langle a''|A|a'\rangle\langle a'|\alpha\rangle$$

$$= \sum_{a'} \underbrace{a'}_{\text{measured value}} \underbrace{|\langle a'|\alpha\rangle|^2}_{\text{probability}}$$

hence eigenvalues $\neq$ expectation values

i.e the expectation value of S_z can be anything between $-\hbar/2$ to $\hbar/2$ ($0.273\hbar$?) but the eigenvalue assumes only two values $\hbar/2$ and $-\hbar/2$

→ Selective measurement (filtration) selects only one of the eigenkets

$$\Lambda_{a'}|\alpha\rangle = |a'\rangle\langle a'|\alpha\rangle \quad \text{(selective measurement)}$$

The uncertainity relation

Given an observable A, define an operator

$$\Delta A \equiv A - \langle A\rangle$$

$(\Delta A)^2$ is known as the dispersion of A

$$\langle(\Delta A)^2\rangle = \langle(A^2 - 2A\langle A\rangle + \langle A\rangle^2)\rangle = \langle A^2\rangle - \langle A\rangle^2$$

For any state we must have

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geqslant \frac{1}{4} |\langle [A,B] \rangle|^2$$

where $[A,B]$ is the commutator

$$[A,B] = AB - BA$$

$\{A,B\}$ is the anti-commutator

$$\{A,B\} = AB + BA$$

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