343 C3 P1 V1 ....../....../.......

Quantum Channels and Open Quantum Systems

In practice many external processes influence pure quantum states in such a way that the formalism of pure quantum states is not adequate

"Closed quantum system" → Pure quantum states

"Open quantum system" → Coupling of pure states with surroundings

In general, a (mixed) quantum state ρ_{in} can undergo a set of unitary and non-unitary transformations due to error processes and coupling with environment. These processes will result in a density matrix ρ_{out}

$$\rho_{in} \rightarrow \text{Quantum Channel (Quantum System)} \rightarrow \rho_{out}$$

This is a linear map from a density matrix to density matrix, however there are some restrictions

Channels as linear maps

A quantum system (channel) can be represented by a linear map Λ

ρ_{in} : $n \times n$ input

ρ_{out} : $m \times m$ output

$$\rho_{out} = \Delta(\rho_{in})$$

$$\Delta : \mathcal{M}_{n \times n} \longrightarrow \mathcal{M}_{m \times m}$$

The map is not necessarily isomorphism, only for unitary operators

Since both the input and output matrices are <u>physical</u>, there are some constraints on Δ

- Any physical density matrix should be positive semi-definitive (i.e. should have only non-negative eigenvalues)

for any extended state ρ_{ext}

$$(\Delta \otimes I_{n_2}) : \mathcal{M}_{n_1 \times n_1} \otimes \mathcal{M}_{n_2 \times n_2} \longrightarrow \mathcal{M}_{m_1 \times m_1} \otimes \mathcal{M}_{m_2 \times m_2}$$

$$(\Delta \otimes I_{n_2})(\rho_{ext,in}) = \rho_{ext,out} \geq 0$$

completely positive property

$\mathcal{M}_{n_1 \times n_1}$ to $\underbrace{\mathcal{M}_{n_1 \times n_1}}_{\text{system subspace}} \otimes \underbrace{\mathcal{M}_{n_2 \times n_2}}_{\text{environment subspace}}$ can be viewed as coupling to environment

- Any physical state ρ should have unit trace for normalization

$$\mathrm{tr}(\Lambda(\rho)) = \mathrm{tr}(\rho) \quad \forall \rho$$

trace preserving property

Any map that is completely positive (CP) and trace preserving (TP) is called a CPTP-map
Usually $m=n$ in our context

Representations of Λ

There are various representations of a channel

1. The χ matrix

$$\Lambda(\rho) = \sum_{m,n=0}^{d^2-1} \chi_{m,n} B_m \rho B_n^\dagger$$

Process matrix

Basis

χ is a $d^2 \times d^2$ matrix called the process matrix
χ together with the basis $\{B_i\}$ completely characterizes Λ. Often $\{B_i\}$ is Pauli basis.

For an unitary process χ can be calculated as

Paulis

$$U = \sum_{P_k \in P_n} \rho_k P_k \qquad \rho_k = \langle P_k, U \rangle = \mathrm{tr}[P_k^\dagger U]$$

$$\Lambda(\rho) = U \rho U^\dagger = \sum_{P_m P_n \in P_n} \rho_m \rho_n^* P_m \rho P_n^\dagger$$

......./......./.......

$$X = |\mathcal{U}\rangle_p \langle\mathcal{U}|_p$$

$|\mathcal{U}\rangle_p$ the vector of all weights of Pauli decomposition of $\mathcal{U}$

2. Choi Matrix

For Δ acting on n qubits, the Choi matrix ρ_{choi} is the density matrix obtained after putting half of the maximally entangled state $|\Omega\rangle$ through the channel Δ while doing nothing on the other half.

$$d = 2^n$$

$$\rho_{choi} = (\Lambda \otimes I)(|\Omega\rangle\langle\Omega|) =$$

$$= \sum_{i,j} \frac{1}{d} \Lambda(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

Straightforward explanation: ρ_{choi} is a block matrix with at (i,j)-th block you have $\Lambda(|i\rangle\langle j|)$. By linearity combining the blocks will give you the action on ρ_{in}

3. Kraus decomposition

Any linear map $\Delta(\rho)$ can be written as $\Lambda(\rho) = \sum_k L_k \rho R_k^\dagger$ for some set of operators $\{L_k\}$ and $\{R_k\}$ by summing over enough k. The Kraus representation is when $\{L_k\} = \{R_k\}$. These are then called Kraus

GÜLPAŞ

operators $\{A_i\}$

$$\Lambda(\rho) = \sum_i A_i \rho A_i^\dagger$$

This representation is useful: for every A_k there is a probability $tr[A_k \rho A_k^\dagger]$ that it will happen, and therefore the resulting state is a statistical mixture of $A_k \rho A_k^\dagger$ states

However $\{A_k\}$ is not unique. Multiple sets of operators represent the same channel any two sets of Kraus operators $\{A_k\}$ and $\{B_k\}$ that correspond to the same channel are linked via a unitary transformation

$$B_i = \sum_j U_{ij} A_j$$

4. Superoperator representation

Vectorisation of χ matrix gives the superoperator representation

$$|ABC\rangle\rangle = (C^T \otimes A)|B\rangle\rangle$$

$$|\rho_{out}\rangle\rangle = S|\rho_{in}\rangle\rangle = \left(\sum_{m,n} \chi_{mn} \bar{B}_n \otimes B_m\right)|\rho_{in}\rangle\rangle$$

$$|\Lambda(\rho)\rangle\rangle = \sum_k (R_k^T \otimes L_k)|\rho\rangle\rangle \hat{=} \sum_{m,n,i,j} A_{mn\,ij}\, \rho_{ij}$$

This representation simplifies representing action of multiple maps i.e. $S = S_1 S_2$ as a single matrix

Relations between channel representations

$$\Lambda(\rho) = d\, tr[\rho_{choi}(I \otimes \rho^{T})]$$

$$\rho_{choi} = \sum_{m,n} \chi_{m,n} |B_m\rangle\rangle\langle\langle B_n|$$

$$\chi_{m,n} = \langle\langle B_m | \rho_{choi} | B_n \rangle\rangle$$

$$\chi_{m,n} = \sum_k \langle B_m, A_k\rangle\langle A_k, B_n\rangle = \sum_k tr[B_m^{\dagger} A_k]\, tr[B_n^{\dagger} A_k]^*$$

$$\rho_{choi} = \frac{1}{d} \sum_{i,j,k} A_k |i\rangle\langle j| A_k^{\dagger} \otimes |i\rangle\langle j|$$

$$S = \sum \chi_{m,n} \overline{B_n} \otimes B_m$$

$$\chi_{m,n} = tr[(\overline{B_n} \otimes B_m)^{\dagger} S]$$

Common channels

Unitary Channels

A set of interacting qubits being added on can be represented by the channel

$$\Delta(\rho) = U \rho U^{\dagger}$$

Pauli channels

When $U \to$ Paulis $\Delta(\rho) = \sum_{P_k \in P_n} p_k P_k \rho P_k$

with $\sum_k p_k = 1$

GÜLPAŞ

Partial trace

Taking the partial trace of a system can be viewed as a channel. Tracing out the i^{th} qubit of an n-qubit system has Kraus op.

$$A_k = \underbrace{I^{\otimes i-1}}_{\text{identity } i-1 \text{ times}} \otimes \underbrace{\langle k|}_{\text{computational basis}} \otimes I^{\otimes n-i}$$

$$M_{d\times d} \to M_{(d-1)\times(d-1)}$$

Common Error channels

Systematic errors

Implementation of a gate U_{act} can deviate from the ideal by a constant operation U_{err}

$$U_{act} = U_{err} U_{ideal}$$

normally, this error is small

$$U_{err} \simeq I$$

they usually mean poor calibration

Dephasing channel

When there is a probability p that the (relative) phase of a qubit flips, the phase becomes gradually less defined this is called dephasing error. Kraus operators for this channel is

$$A_1 = \sqrt{1-p}\, I \quad A_2 = \sqrt{p}\, Z$$

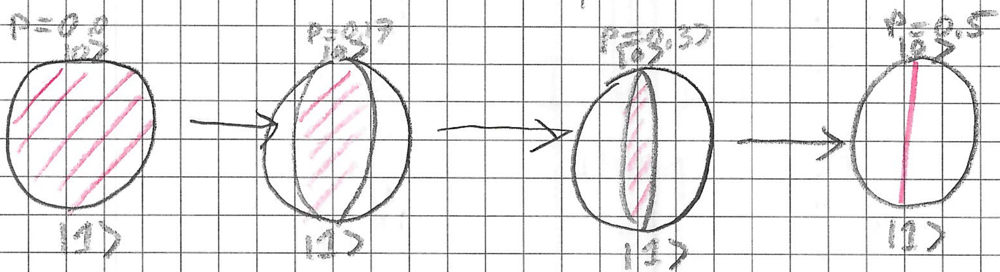
consider a qubit in the state $\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$

$$\Lambda_{deph(p)}\left(\begin{bmatrix} a & b \\ b^* & d \end{bmatrix}\right) = (1-p)\begin{bmatrix} a & b \\ b^* & d \end{bmatrix} + p\begin{bmatrix} a & -b \\ -b^* & d \end{bmatrix}$$

$$= \begin{bmatrix} a & (1-2p)b \\ (1-2p)b^* & d \end{bmatrix}$$

Pure states that are superpositions have off diagonal elements, where as purely statistical mixtures have no off diagonal elements, dephasing destroys coherence between the two basis states of the qubit

When $p = 1/2$ the channel $\Lambda_{deph(1/2)}$ is known as the complete dephasing channel
→ It decoheres the qubit



A qubit may decohere gradually with rate T_2 as e^{-t/T_2}

343 C3 PSV1 ....../....../.......

Depolarizing Channel

In the depolarizing channel, the Pauli X, Y and Z operators are all applied to the state with equal probability $p/3$, while the state can also be left intact. For single qubit

$$\{A_k\} = \left\{ \sqrt{1-p}\, I, \sqrt{\frac{p}{3}}X, \sqrt{\frac{p}{3}}Y, \sqrt{\frac{p}{3}}Z \right\}$$

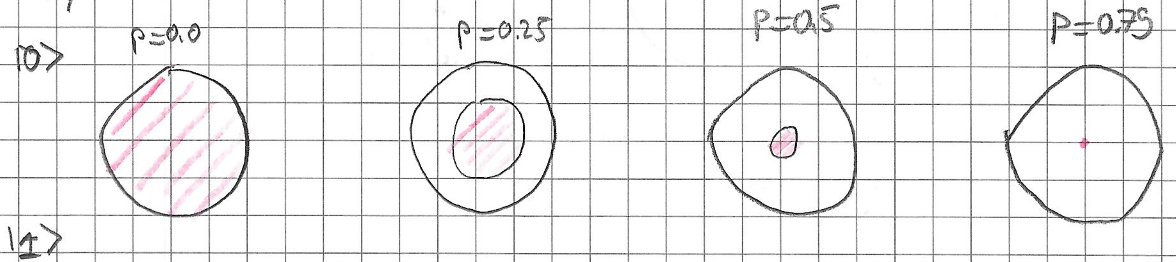
$\Lambda_{dep(p)}$ acting on a qubit in state $\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$ yields

$$\Lambda_{dep(p)}\left(\begin{bmatrix} a & b \\ b^* & d \end{bmatrix}\right) = \begin{bmatrix} (1-\frac{2p}{3})a + \frac{2p}{3}d & (1-\frac{4p}{3})b \\ (1-\frac{4p}{3})b^* & (1-\frac{2p}{3})d + \frac{2p}{3}a \end{bmatrix}$$

$$= (1-\frac{4p}{3})\rho + \frac{4p}{3}\frac{I}{2}$$

The depolarizing channel takes a convex combination of ρ with the maximally mixed state, regardless of the initial state ρ.

This can be visualised as a deflating Bloch sphere



Amplitude damping channel

The amplitude damping channel gradually maps a qubit to the $|0\rangle$ state

There are 2 Kraus operators for this channel

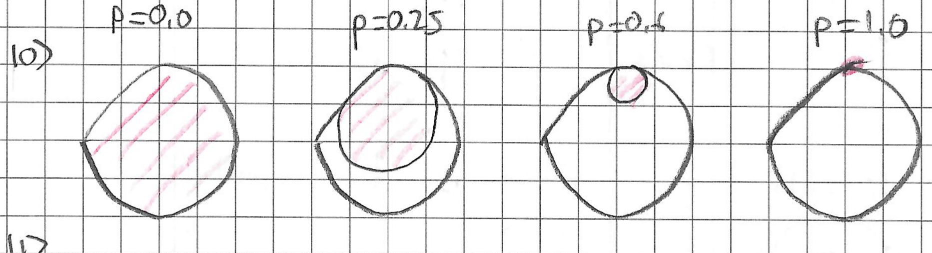
$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} = \frac{1+\sqrt{1-p}}{2} I + \frac{1-\sqrt{1-p}}{2} Z$$

$$A_2 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} = \sqrt{p}\,(X+iY)$$

A_2 is the lowering operator, that maps $|1\rangle$ state to $|0\rangle$ state

$$\Delta_{amp}\left(\begin{bmatrix} a & b \\ b^* & d \end{bmatrix}\right) = \begin{bmatrix} a+pd & \sqrt{1-p}\,b \\ \sqrt{1-p}\,b^* & (1-p)d \end{bmatrix}$$

Amplitude damping also deflates the Bloch sphere, however $|0\rangle$ is untouched



The characteristic time that is associated with amplitude damping is known as the relaxation time T_1 corresponding to decoherence time T_2. If a state relaxes, it also gradually looses coherence

The amplitude damping model with 2 Kraus operators is only valid for $T=0$. At finite temperature, there will be a statistical occupation of $|2\rangle$ channel, hence 3 Kraus operators are required

Process Fidelity

A measure that is used to test the quality of a map is process fidelity

If there is a desired unitary process χ_{ideal} then the process fidelity of χ is

$$F_p = tr[\chi_{ideal}\chi] = tr[\chi\chi_{ideal}] = tr[\rho_{choi}\rho_{choi,ideal}]$$

If everything is diagonalised using the unitary basis of χ_{ideal}, $|u\rangle$

$$F_p = \langle u|\chi|u\rangle$$